ON BERNOULLI CONVOLUTIONS AND THE PROJECTION OF ERGODIC MEASURES

OR LANDESBERG

Adviser: Prof. Elon Lindenstrauss.

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Abstract. This paper is concerned with the question of absolute continuity of distributions μ_λ of random series $\sum \pm\lambda^n$ given as a projection of shift-ergodic probability measures μ on the sequence space $\{\pm 1\}^{\mathbb{N}}$ and the answer's dependence upon $\lambda \in \left(\frac{1}{2}, 1\right)$. In [8], Y. Peres and B. Solomyak proved that given a shift-ergodic probability measure μ on $\{\pm 1\}^{\mathbb{N}}$ with Kolmogorov-Sinai entropy h, its projection μ_λ is absolutely continuous for *Leb*-a.e. $\lambda \in \left(2^{-h}, \alpha\right)$, where $\alpha \approx 0.668475$. It is conjectured that this is true for *Leb*-a.e. $\lambda \in \left(2^{-h}, 1\right)$. Employing the techniques developed by Solomyak and Peres along with a decomposition of μ allows significantly extending the area of almost-sure absolute continuity for measures with high entropy. In Particular, the conjecture is confirmed for Markov measures satisfying some conditional entropy bounds, such as the Markov measures given by marginal $\begin{pmatrix} p & 1-p \\ 1-p & p \end{pmatrix}$ for any $p \in [0.433, 0567]$.

In addition, general properties of the projection of ergodic measures are established - Law of pure types and the set of λ's corresponding to singular measures being G_δ.

Contents

1. Background and Historical notes

1.I. **Setting and Background.** Consider the sequence space $\Omega = \{\pm 1\}^{\mathbb{N}}$ equipped with the shift map $\sigma : (\omega_1, \omega_{2,\ldots}) \mapsto (\omega_2, \omega_{3,\ldots})$ and the metric $d(\omega, \tau) = 2^{-|\omega\wedge\tau|}$, $|\omega \wedge \tau| = \min_k \{\omega_k \neq \tau_k\}$. Given a parameter $\lambda \in (0, 1)$ we define the projection

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map $\pi_\lambda : \Omega \to \mathbb{R}$ by:

$$\pi_\lambda(\omega) = \sum_{n=0}^{\infty} \omega_n \lambda^n$$

This map is clearly continuous and measurable. Given some measure μ on Ω we denote its projection by $\mu_\lambda = \pi_\lambda \mu$. We are concerned with the question -
For which μ and λ is μ_λ absolutely continuous with respect to Lebesgue measure $\mathcal{L}$?

A first answer can be given by considering the geometry of suppμ_λ. By definition, the projected measure is supported on $\pi_\lambda(\Omega)$ which can be viewed as the attractor of the IFS $\Phi_\lambda = \{\varphi_-^\lambda, \varphi_+^\lambda\}$ with $\varphi_\pm^\lambda(x) = \lambda x \pm 1$, since:

$$\varphi_\pm^\lambda(\pi_\lambda(\omega)) = \pm 1 + \sum_{n=0}^{\infty} \omega_n \lambda^{n+1}$$

and consequently:

$$\varphi_-^\lambda(\pi_\lambda(\Omega)) \cup \varphi_+^\lambda(\pi_\lambda(\Omega)) = \pi_\lambda([-1]) \cup \pi_\lambda([+1]) = \pi_\lambda(\Omega)$$

where we use the notation $[i]$, for any $i \in \{\pm 1\}^k$, to represent the corresponding cylinder set $[i] = \{\omega \in \Omega \,|\, \omega_1\omega_2...\omega_k = i\}$.
In the case where $\lambda \in \left(0, \frac{1}{2}\right)$, the IFS Φ_λ and its attractor satisfy:

$$\varphi_-^\lambda(\pi_\lambda(\Omega)) \cap \varphi_+^\lambda(\pi_\lambda(\Omega)) = \emptyset$$

a condition called strong separation. This condition implies the Hausdorff dimension of $\pi_\lambda(\Omega)$ is equal to the similarity dimension of $\Phi_\lambda = \frac{-1}{\log_2 \lambda} < 1$ (see theorem 5.16 in [4]), meaning μ is supported on a set of zero Lebesgue measure. Therefore, all measures on Ω will project onto singular measures by π_λ for any $\lambda \in \left(0, \frac{1}{2}\right)$.
The question remains - what happens when $\lambda \in \left[\frac{1}{2}, 1\right)$ and $\pi_\lambda(\Omega) = \left[\frac{-1}{1-\lambda}, \frac{1}{1-\lambda}\right]$?

1.II. **Bernoulli Convolutions**[1]**.** The case where μ is taken to be the Bernoulli measure $\nu^{\frac{1}{2}} = \left(\frac{1}{2}, \frac{1}{2}\right)^{\mathbb{N}}$ on Ω has been fruitfully studied since the 1930's. In this case $\nu_\lambda^{\frac{1}{2}}$ is the infinite convolution of the measures $\frac{1}{2}(\delta_{-\lambda^n} + \delta_{\lambda^n})$, hence the name 'Infinite Bernoulli Convolutions'. Denote by $S_\perp$ the set of $\lambda \in \left(\frac{1}{2}, 1\right)$ for which $\nu_\lambda^{\frac{1}{2}} = \pi_\lambda \nu^{\frac{1}{2}}$ is singular. The only elements known to be found in $S_\perp$ are reciprocals of Pisot numbers in $(1, 2)$[2]. The proof is due to Erdös (1939) using harmonic analysis. It is conjectured that these are the only elements of $S_\perp$. The first important result in that direction is also due to Erdös (1940) where he proved that $S_\perp \cap (a, 1)$ has zero Lebesgue measure for some $a < 1$. Kahane later indicated the argument actually implies that the Hausdorff dimension of $S_\perp \cap (a, 1)$ tends to 0 as $a \nearrow 1$. In [11], Boris Solomyak showed $S_\perp$ is of zero Lebesgue measure using a certain transversality property of the family of functions $\mathcal{F} = \left\{f(x) = \sum_{k=0}^{\infty} a_k x^k \,|\, a_k \in \{\pm 1, 0\}\right\}$ and a sub-family. Solomyak together with Yuval Peres later published a simpler proof [7]. Recently Pablo Shmerkin[9], relying on work by Michael Hochman[3], proved the set $S_\perp$ is actually of Hausdorff dimension 0, the strongest result yet.
Both the Erdös-Kahane and the Hochman-Shmerkin approaches rely heavily upon the infinite convolution structure of $\nu_\lambda^{\frac{1}{2}}$, something one cannot assume when dealing

[1]The historical background goes along the lines of [6] with some recent updates.
[2]Those algebraic numbers whose Galois conjugates are of modulus< 1

with the projection of a general ergodic measure. This paper will employ the techniques developed by Peres and Solomyak.

1.III. **General Ergodic Measures.** In [8], Peres and Solomyak effectively proved the following theorem (in a much broader context):

Theorem. *Given a σ-ergodic probability measure μ on Ω, μ_λ is:*

(1) *absolutely continuous for $\mathcal{L}$-a.e.* $\lambda \in \left(2^{-h_\mu(\sigma)}, 0.668475\right)$
(2) *singular for all* $\lambda < 2^{-h_\mu(\sigma)}$

The value 0.668475 is due to the transversality property. This property will be discussed in detail in section 5. We will give the proof of claim 2 here and deduce claim 1 later, as a consequence of theorem 3.2.

Proof. Using the Shannon-McMillan-Breiman theorem we know that for μ-a.e. $\omega \in \Omega$:

$$\lim_{n\to\infty} \frac{-1}{n} \log\left(\mu\left([\omega]_n\right)\right) = h_\mu(\sigma)$$

where $[\omega]_n = [\omega_1 ... \omega_n]$. Hence by Billingsley's lemma the Hausdorff dimension of μ in Ω is equal to $h_\mu(\sigma)$.
Notice that the map π_λ is $(-\log\lambda) - \textit{Hölder}$ since:

$$|\pi_\lambda(\omega) - \pi_\lambda(\tau)| \leq C\lambda^{-|\omega\wedge\tau|} \leq C\left(d(\omega,\tau)\right)^{-\log\lambda}$$

Using this fact we receive:

$$\dim_{\mathcal{H}} \mu_\lambda \leq \frac{-1}{\log\lambda} \dim_{\mathcal{H}} \mu = -\frac{h_\mu(\sigma)}{\log\lambda}$$

When $\lambda < 2^{-h_\mu(\sigma)}$ we have $\dim_{\mathcal{H}} \mu_\lambda < 1$ and consequently μ_λ is singular. □

It is naturally conjectured that:

Conjecture. *Given a σ-ergodic probability measure μ, its projection μ_λ is absolutely continuous for $\mathcal{L}$-a.e.* $\lambda \in \left(2^{-h_\mu(\sigma)}, 1\right)$.

In [8] the authors tackle this conjecture for the biased Bernoulli convolutions, where μ is taken to be the Bernoulli measure $\nu^p = (p, 1-p)^{\mathbb{N}}$ for some $p \in (0,1)$, and prove the following theorem:

Theorem. ν_λ^p *is absolutely continuous for $\mathcal{L}$-a.e.* $\lambda \in \left(p^p(1-p)^{1-p}, 1\right)$, *for any* $p \in \left[\frac{1}{3}, \frac{2}{3}\right]$, [3]

2. Law of Pure Types

In question of absolute continuity, a measure is said to be of *pure type* if its Lebesgue decomposition with respect to $\mathcal{L}$ is trivial, i.e. it is either absolutely continuous or singular. Jessen and Wintner (1935) showed that any convergent infinite convolution of discrete measures is of pure type. In [6] is given a proof that any self-similar probability measure on $\mathbb{R}^d$ is of pure type. We give two proofs to the following proposition:

[3]Note that $h_{\nu^p}(\sigma) = H(p, 1-p) = -p\log p - (1-p)\log(1-p)$

Proposition 2.1. *Given a σ-ergodic probability measure μ on Ω and some $\lambda \in (0,1)$, the projected measure μ_λ is of pure type with respect to Lebesgue measure, i.e. $\mu_\lambda \ll \mathcal{L}$ or $\mu_\lambda \perp \mathcal{L}$.*

The following proofs can be extended to suit a wider variety of IFS symbols space projections and can also be adapted to proving pure type with respect to any α-dimensional Hausdorff measure, as been done in 6.I.

Elementary Proof.

Proof. Denote $\Phi_\lambda = \left\{\varphi_-^\lambda, \varphi_+^\lambda\right\}$ with $\varphi_\pm^\lambda(x) = \lambda x \pm 1$. assume there exists a set $A \subseteq \mathbb{R}$ with $\mu_\lambda(A) > 0$ and $\mathcal{L}(A) = 0$. For every finite sequence $i \in \{\pm 1\}^n$ the map $\varphi_{i_1}^\lambda \circ ... \circ \varphi_{i_n}^\lambda$ is affine thus giving $\mathcal{L}\left(\left(\varphi_{i_1}^\lambda \circ ... \circ \varphi_{i_n}^\lambda\right)(A)\right) = 0$ and consequently:

$$\mathcal{L}\left(\bigcup_n \bigcup_{i\in\{-1,1\}^n} \left(\varphi_{i_1}^\lambda \circ ... \circ \varphi_{i_n}^\lambda\right)(A)\right) = 0$$

On the other hand:

$$\bigcup_n \bigcup_{i\in\{-1,1\}^n} \left(\varphi_{i_1}^\lambda \circ ... \circ \varphi_{i_n}^\lambda\right)(A) = \pi_\lambda\left(\bigcup_n \sigma^{-n}\left(\pi_\lambda^{-1} A\right)\right) = A'$$

Since $\pi_\lambda^{-1} A \subseteq \Omega$ is a set of positive μ-measure, by ergodicity:

$$\mu\left(\bigcup_n \sigma^{-n}\left(\pi_\lambda^{-1} A\right)\right) = 1$$

meaning $\mu_\lambda(A') = 1$ and $\mathcal{L}(A') = 0$. □

Sketch of Proof Using the Density Function.

This proof is due to Michael Hochman.

Definition. *The upper 1-dimensional density of a measure ν at $x \in \mathbb{R}^d$ is:*

$$D_1^+(\nu, x) = \limsup_{r\to 0} \frac{\nu(B_r(x))}{2r}$$

where $B_r(x)$ is the closed ball of radius r around x.

Denote:

$$A_\infty^1 = \left\{\omega \in \Omega \,\middle|\, D_1^+(\mu_\lambda, \pi_\lambda(\omega)) < \infty\right\} = \pi_\lambda^{-1} \circ \left(D_1^+\right)^{-1}([0,\infty))$$

Sketch of proof: Using the affine nature of $\varphi_\pm^\lambda$ and the Lebesgue-Besicovitch density theorem (see 2.14 in [5]) it can be shown that for μ-a.e. $\omega \in \Omega$:

$$D_1^+(\mu_\lambda, \pi_\lambda\omega) \geq C_+(\omega) \cdot D_1^+(\mu_\lambda, \pi_\lambda(1\omega))$$

$$D_1^+(\mu_\lambda, \pi_\lambda\omega) \geq C_-(\omega) \cdot D_1^+(\mu_\lambda, \pi_\lambda(-1\omega))$$

where the functions C_+, C_- are positive μ-a.e., proving $\sigma^{-1}A^1_\infty \underset{\mu}{=} A^1_\infty$. The ergodicity of μ implies either:

$$\mu\left(A^1_\infty\right) = 0 \implies \mu_\lambda \perp \mathcal{L}$$

or:

$$\mu\left(A^1_\infty\right) = 1 \implies \mu_\lambda \ll \mathcal{L}$$

$\square$

A full proof is given in appendix 6.I

3. Entropy and Absolute Continuity

We begin the main proof with some notations: $\mathcal{P} = \{[-1], [1]\}$ is the generating partition for $(\Omega, \mathscr{A}, \sigma)$ and $\mathscr{A}_n = \bigvee_{i=0}^{n-1} \sigma^{-i}\mathcal{P}$. Given a set of indices $E \subseteq \mathbb{N}$ we denote:

$$\mathscr{A}_E = \bigvee_{i \in E} \sigma^{-i}\mathcal{P}$$

the σ-algebra controlling all the E-indices and:

$$\mathcal{F}_E = \left\{ f(x) = \sum_{k=0}^{\infty} a_k x^k \in \mathcal{F} \,|\, \forall k \in E \quad a_k = 0 \right\}$$

the corresponding sub-family of:

$$\mathcal{F} = \left\{ f(x) = \sum_{k=0}^{\infty} a_k x^k \,|\, a_k \in \{\pm 1, 0\} \right\}$$

with all E-indices set to 0.
E is said to be N_0-periodic if $\forall i \in \mathbb{N} \quad i \in E \iff i + N_0 \in E$. We refer to the empty set $\emptyset$ as 1-periodic, with $\mathcal{F}_\emptyset = \mathcal{F}$.

Definition 3.1. Given a set of indices $E \subseteq \mathbb{N}$, we say the interval $I \subseteq \left[\frac{1}{2}, 1\right]$ is an interval of *δ-transversality for the family of functions* $\mathcal{F}_E$ if for any sub-interval $I_0 = [\lambda_0, \lambda_1] \subseteq I$ and all $\phi \in \mathcal{F}_E$ and $r > 0$:

$$\mathcal{L}\left\{x \in I_0 \,|\, |\phi(x)| \le r\right\} \le 2\delta^{-1}\lambda_0^{-\wedge(\phi)} r$$

where we denote $\wedge\left(\sum_{k=0}^{\infty} a_k x^k\right) = \inf_k (a_k \neq 0) \in \mathbb{N} \cup \{\infty\}$.

Remark. This definition of δ-transversality is different than the one used by Peres and Solomyak in [7]. This is only in order to postpone some technicalities arising from fixing the E-indices to section 5 where we show how the original condition of δ-transversality implies our definition.

3.I. Main Theorem.

Theorem 3.2. *Let $E \subseteq \mathbb{N}$ be an N_0-periodic set of indices with I an interval of δ-transversality for the family $\mathcal{F}_E$. Let μ be a probability measure on Ω with:*

$$\lim_{l \to \infty} \frac{-1}{l \cdot N_0} \log\left(\mu_\omega^{\mathscr{A}_E}\left([\omega]_{l \cdot N_0}\right)\right) \ge \alpha$$

for μ-a.e. $\omega \in \Omega$ where $\mu = \int \mu_\omega^{\mathscr{A}_E} d\mu(\omega)$ is the decomposition of μ with respect to the σ-algebra $\mathscr{A}_E$. Then μ_λ is absolutely continuous for $\mathcal{L}$-a.e. $\lambda \in (2^{-\alpha}, 1) \cap I$.

Proof. We will use a decomposition of μ_λ induced by the decomposition of μ:

$$\mu_\lambda = \int_\Omega \mu_\lambda^{\mathscr{A}_E,\omega} d\mu(\omega)$$

where $\mu_\lambda^{\mathscr{A}_E,\omega} = \pi_\lambda \mu_\omega^{\mathscr{A}_E}$. If we show that for $\mathcal{L}$-a.e. $\lambda \in I_h = \left(2^{-h}, 1\right) \cap I$ the measure $\mu_\lambda^{\mathscr{A}_E,\omega}$ is absolutely continuous for μ-a.e. $\omega \in \Omega$ then for all such λ the measure μ_λ is also absolutely continuous. Consider the function:

$$D_\mu(\lambda, \omega, \tau) = \liminf_{r \searrow 0} \frac{\mu_\lambda^{\mathscr{A}_E,\omega}\left(B_r\left(\pi_\lambda(\tau)\right)\right)}{2r}$$

For any fixed λ and ω, $D_\mu(\lambda, \omega, \cdot)$ is the lower density function of $\mu_\lambda^{\mathscr{A}_E,\omega}$. The measure $\mu_\lambda^{\mathscr{A}_E,\omega}$ is absolutely continuous if and only if $D_\mu(\lambda, \omega, \cdot) < \infty$ $\mu_\lambda^{\mathscr{A}_E,\omega}$-a.e. (see theorem 2.12 in [5]). Therefore it would suffice to show that for $\mathcal{L}$-a.e. $\lambda \in I_h$ the function $D_\mu(\lambda, \omega, \tau)$ receives finite value for μ-a.e. ω and $\mu_\lambda^{\mathscr{A}_E,\omega}$-a.e. $\tau \in \Omega$. Using Fubini's theorem and the measurability of $D_\mu(\lambda, \omega, \tau) : I_h \times \Omega \times \Omega \to \mathbb{R}^+ \cup \{\infty\}$ (established in lemma 6.6) we can reduce to proving that for μ-a.e. ω:

$$D_\mu(\lambda, \omega, \tau) < \infty$$

for $\mu_\lambda^{\mathscr{A}_E,\omega}$-a.e. τ and $\mathcal{L}$-a.e. $\lambda \in I_h$. As assumed:

$$\lim_{l \to \infty} \frac{-1}{l \cdot N_0} \log\left(\mu_\omega^{\mathscr{A}_E}\left([\omega]_{l \cdot N_0}\right)\right) \geq \alpha \tag{3.1}$$

for all ω in a set Ω' of full μ-measure. For μ-a.e. ω, $\mu_\omega^{\mathscr{A}_E}(\Omega') = 1$ meaning there exists a set Ω'' of full μ-measure for which all ω in Ω'' , satisfy property 3.1 for $\mu_\omega^{\mathscr{A}_E}$-a.e. τ. Fix such an $\omega \in \Omega''$.
Using Egoroff's theorem there exists a sequence of sets $A_n \subseteq \Omega$ with:

$$\mu_\omega^{\mathscr{A}_E}\left(\bigcup_{n=1}^{\infty} A_n\right) = 1$$

for which the convergence in (3.1) is uniform[4]. Assuming $I_h \neq \emptyset$ we denote $\lambda_0 = \inf I_h$ and for all $0 < \varepsilon < |I_h|$, $\lambda_{0,\varepsilon} = \lambda_0 + \varepsilon$ and $I_h^\varepsilon = I_h \cap [\lambda_{0,\varepsilon}, 1]$. Using Fatou's lemma and Fubini's theorem we calculate:

$$\int_{I_h^\varepsilon} \int_{A_n} D_\mu(\lambda, \omega, \tau)\, d\mu_\omega^{\mathscr{A}_E}(\tau)\, d\lambda \leq \liminf_{r \searrow 0} \frac{1}{2r} \int_{I_h^\varepsilon} \int_{A_n} \mu_\lambda^{\mathscr{A}_E,\omega}\left(B_r\left(\pi_\lambda(\tau)\right)\right) d\mu_\omega^{\mathscr{A}_E}(\tau)\, d\lambda =$$

$$= \liminf_{r \searrow 0} \frac{1}{2r} \int_{I_h^\varepsilon} \int_{A_n} \int_\Omega \chi_{\left\{(\tau,\tau') \,\middle|\, |\pi_\lambda(\tau) - \pi_\lambda(\tau')| \leq r\right\}} d\mu_\omega^{\mathscr{A}_E}(\tau')\, d\mu_\omega^{\mathscr{A}_E}(\tau)\, d\lambda =$$

$$= \liminf_{r \searrow 0} \frac{1}{2r} \int_{A_n} \int_\Omega \mathcal{L}\left\{\lambda \in I_h^\varepsilon \,\middle|\, |\pi_\lambda(\tau) - \pi_\lambda(\tau')| \leq r\right\} d\mu_\omega^{\mathscr{A}_E}(\tau')\, d\mu_\omega^{\mathscr{A}_E}(\tau) = (\star)$$

Recall at this point that the measure $\mu_\omega^{\mathscr{A}_E}$ is supported on $[\omega]_{\mathscr{A}_E}$, the ω-atom with respect to $\mathscr{A}_E$[5], meaning that for $\mu_\omega^{\mathscr{A}_E}$-a.e. τ $\forall k \in E$ $\tau_k = \omega_k$ which consequently

[4]meaning that $\forall \beta < \alpha\ \exists k\ \forall l > k\ \ \frac{-1}{l \cdot N_0} \log\left(\mu_\omega^{\mathscr{A}_E}\left([\omega]_{l \cdot N_0}\right)\right) > \beta$

[5]$\mathscr{A}_E$ is clearly a countably generated σ-algebra

assures $\frac{1}{2}\left(\pi_\lambda\left(\tau\right)-\pi_\lambda\left(\tau'\right)\right)\in\mathcal{F}_E$. Due to the δ-transversality on I we insert:

$$\mathcal{L}\left\{\lambda\in I_h^\varepsilon\,\middle|\,\left|\pi_\lambda\left(\tau\right)-\pi_\lambda\left(\tau'\right)\right|\le r\right\}=$$
$$=\mathcal{L}\left\{\lambda\in I_h^\varepsilon\,\middle|\,\left|\frac{1}{2}\left(\pi_\lambda\left(\tau\right)-\pi_\lambda\left(\tau'\right)\right)\right|\le\frac{r}{2}\right\}\le\delta^{-1}\lambda_{0,\varepsilon}^{-\wedge(\phi)}r$$

to conclude:

$$\begin{aligned}(\star)\quad &\le (2\delta)^{-1}\int_{A_n}\int_\Omega\lambda_{0,\varepsilon}^{-\wedge\left(\pi_\lambda(\tau)-\pi_\lambda(\tau')\right)}d\mu_\omega^{\mathscr{A}_E}\left(\tau'\right)d\mu_\omega^{\mathscr{A}_E}\left(\tau\right)=\\ &= (2\delta)^{-1}\int_{A_n}\int_\Omega\lambda_{0,\varepsilon}^{-\left|\tau\wedge\tau'\right|}d\mu_\omega^{\mathscr{A}_E}\left(\tau'\right)d\mu_\omega^{\mathscr{A}_E}\left(\tau\right)=\\ &= (2\delta)^{-1}\int_{A_n}\sum_{l=0}^{\infty}\lambda_{0,\varepsilon}^{-l}\mu_\omega^{\mathscr{A}_E}\left(\left[\tau\right]_l\right)d\mu_\omega^{\mathscr{A}_E}\left(\tau\right)\le\\ &\le (2\delta)^{-1}\left(\sum_{\substack{s\in E^c\\ s<N_0}}\lambda_{0,\varepsilon}^{-s}\right)\int_{A_n}\sum_{l=0}^{\infty}\lambda_{0,\varepsilon}^{-l\cdot N_0}\mu_\omega^{\mathscr{A}_E}\left(\left[\tau\right]_{l\cdot N_0}\right)d\mu_\omega^{\mathscr{A}_E}\left(\tau\right)\end{aligned}$$

By the definition of A_n there exists a k for which all $l>k$ admit:

$$\frac{-1}{l\cdot N_0}\log\left(\mu_\omega^{\mathscr{A}_E}\left(\left[\tau\right]_{l\cdot N_0}\right)\right)>\beta>-\log\lambda_{0,\varepsilon}$$

assuring:

$$\begin{aligned}\sum_{l=0}^{\infty}\lambda_{0,\varepsilon}^{-l\cdot N_0}\mu_\omega^{\mathscr{A}_E}\left(\left[\tau\right]_{l\cdot N_0}\right) &< C+\sum_{l=k+1}^{\infty}\lambda_{0,\varepsilon}^{-l\cdot N_0}2^{-\beta\cdot l\cdot N_0}\\ &= C+\sum_{l=k+1}^{\infty}2^{-l\cdot N_0\left(\beta+\log\lambda_{0,\varepsilon}\right)}<C'<\infty\end{aligned}$$

and consequently:

$$\int_{I_h^\varepsilon}\int_{A_n}D_\mu\left(\lambda,\omega,\tau\right)d\mu_\omega^{\mathscr{A}_E}\left(\tau\right)d\lambda<\infty$$

Taking $\varepsilon\searrow0$ will give $D_\mu\left(\lambda,\omega,\tau\right)<\infty$ for $\mathcal{L}$-a.e. $\lambda\in I_h$ and $\mu_\omega^{\mathscr{A}_E}$-a.e. $\tau\in A_n$. This being true for all n concludes the proof. □

Corollary 3.3. *Let $E\subseteq\mathbb{N}$ be an N_0-periodic set of indices with I an interval of δ-transversality for the family $\mathcal{F}_E$. Let μ be σ-ergodic with $\frac{1}{N_0}h_\mu\left(\sigma^{N_0}|\mathscr{A}_E\right)\ge\alpha$, then the projection μ_λ is absolutely continuous for $\mathcal{L}$-a.e. $\lambda\in\left(2^{-\alpha},1\right)\cap I$.*

Proof. In the case where μ is also σ^{N_0}-ergodic, the conditional Shannon-McMillan-Breiman theorem for the m.p.s. $\left(\Omega,\mu,\sigma^{N_0}\right)$[6] assures the conditions of theorem 3.2 are fulfilled, thus proving the claim.

When μ is not σ^{N_0}-ergodic, we can decompose μ to its ergodic components. Let $A\subseteq\Omega$ be some non-trivial σ^{N_0}-invariant set. μ is σ-ergodic meaning:

$$\mu\left(\bigcup_{k=0}^{\infty}\sigma^{-k}A\right)=1$$

[6]See appendix B in [1]. Notice $\mathscr{A}_E$ is σ^{N_0}-sub-invariant.

since $\sigma^{-N_0}A \subseteq A$ we receive:

$$\mu\left(\bigcup_{k=0}^{N_0-1}\sigma^{-k}A\right)=1$$

and consequently $\mu(A) \geq \frac{1}{N_0}$. This shows that there are a finite number, d, of components in the ergodic decomposition of μ with respect to σ^{N_0}, all of which are supported on sets of the same measure $\frac{1}{d}$. Denote these components by $\mu_1,...,\mu_d$, receiving $\mu=\frac{1}{d}\sum_{i=1}^{d}\mu_i$. Proving the claim for all the μ_i's concludes the proof.
Let $\psi_E:\Omega\to\Omega$ be the map projecting $(\omega_1,\omega_2,...)$ onto its E-indices:

$$\psi_E(\omega_1,\omega_2,...)=(\omega_{e_1},\omega_{e_2},...)$$

where $E=\{e_1,e_2,...\}\subseteq\mathbb{N}$. Denoting $s=|\{e\in E\,|\,e<N_0\}|$, we receive ψ_E is a factor map of dynamical systems:

$$\psi_E:\left(\Omega,\mathscr{A},\sigma^{N_0}\right)\to(\Omega,\mathscr{A},\sigma^s)$$

By definition $\mathscr{A}_E=\psi_E^{-1}\mathscr{A}$ and by the Abramov-Rokhlin formula:

$$h_\mu\left(\sigma^{N_0}|\mathscr{A}_E\right)=h_\mu\left(\sigma^{N_0}\right)-h_{\psi_E\mu}(\sigma^s)$$

The ergodic decomposition of $\psi_E\mu$ with respect to σ^s is $\frac{1}{d}\sum_{i=1}^{d}\psi_E\mu_i$ hence we can decompose the respective entropies[7]:

$$h_\mu\left(\sigma^{N_0}|\mathscr{A}_E\right)=\frac{1}{d}\sum_{i=1}^{d}\left[h_{\mu_i}\left(\sigma^{N_0}\right)-h_{\psi_E\mu_i}(\sigma^s)\right]=\frac{1}{d}\sum_{i=1}^{d}h_{\mu_i}\left(\sigma^{N_0}|\mathscr{A}_E\right)$$

Showing there exists some $1\leq i_0\leq d$ for which:

$$h_{\mu_{i_0}}\left(\sigma^{N_0}|\mathscr{A}_E\right)\geq h_\mu\left(\sigma^{N_0}|\mathscr{A}_E\right)$$

The measure μ_{i_0} is σ^{N_0}-ergodic with $\frac{1}{N_0}h_{\mu_{i_0}}\left(\sigma^{N_0}|\mathscr{A}_E\right)\geq\alpha$ and thus projects as required.
Notice that since μ is σ-invariant and ergodic, the map σ induces a transitive permutation $\Pi_\sigma:\{1,...,d\}\to\{1,...,d\}$ for which $\sigma:\left(\Omega,\mathscr{A},\sigma^{N_0},\mu_i\right)\to\left(\Omega,\mathscr{A},\sigma^{N_0},\mu_{\Pi_\sigma(i)}\right)$ is a factor map of m.p.s. Since entropy only decreases by factorization we receive $h_{\mu_i}\left(\sigma^{N_0}\right)\geq h_{\mu_{\Pi_\sigma(i)}}\left(\sigma^{N_0}\right)$ and transitivity assures that for all $1\leq i,j\leq d$:

$$h_{\mu_i}\left(\sigma^{N_0}\right)=h_{\mu_j}\left(\sigma^{N_0}\right)$$

Let $j\neq i_0$, there exists some k_j with $\mu_j=\sigma^{k_j}\mu_{i_0}$. The N_0-periodicity of E yields the following identity:

$$\sigma^s\circ\psi_{E-k_j}=\sigma^l\circ\psi_E\circ\sigma^{N_0-k_j}$$

where ψ_{E-k_j} is the projection onto the $E-k_j=\{e-k_j\,|\,e\in E\}$ indices and $l=|[e\in E\,|\,e-k_j<0]|$. Therefore:

$$\begin{aligned}h_{\mu_j}\left(\sigma^{N_0}|\mathscr{A}_{E-k_j}\right) &= h_{\mu_j}\left(\sigma^{N_0}\right)-h_{\psi_{E-k_j}\mu_j}(\sigma^s)=\\ &= h_{\mu_{i_0}}\left(\sigma^{N_0}\right)-h_{\left(\sigma^l\circ\psi_E\circ\sigma^{N_0-k_j}\right)\mu_j}(\sigma^s)=(\star)\end{aligned}$$

where we used the fact that $\psi_{E-k_j}\mu_j=\sigma^s\psi_{E-k_j}\mu_j$. Notice that due to the σ^{N_0}-invariance of μ_{i_0}, $\mu_{i_0}=\sigma^{N_0-k_j}\mu_j$ hence:

$$(\star)=h_{\mu_{i_0}}\left(\sigma^{N_0}\right)-h_{\sigma^l\left(\psi_E\mu_{i_0}\right)}(\sigma^s)$$

[7]Theorem 5.27 in [13]

Relying again on the decreasing property of entropy under factorization we receive:

$$h_{\sigma^l\left(\psi_E\mu_{i_0}\right)}\left(\sigma^s\right) \le h_{\psi_E\mu_{i_0}}\left(\sigma^s\right)$$

and consequently:

$$h_{\mu_j}\left(\sigma^{N_0}|\mathscr{A}_{E-k_j}\right) \ge h_{\mu_{i_0}}\left(\sigma^{N_0}|\mathscr{A}_E\right)$$

Notice that I is an interval of $\frac{\delta}{2^{k_j}}$-transversality for the family $\mathcal{F}_{E-k_j}$ thus concluding the proof. □

3.II. **Results for General Measures.** In section 5 we will establish the following results regarding transversality:

- $\left[\frac{1}{2},\alpha_1\right]$ is an interval of δ-transversality for the family $\mathcal{F}$, with $\alpha_1 = 0.668475$.
- $\left[\frac{1}{2},\alpha_2\right]$ is an interval of δ-transversality for the family $\mathcal{F}_{\{i\in\mathbb{N}\,|\,i\equiv 0(mod3)\}}$, with $\alpha_2 = 0.713549$.

Remark. Note that the value of $\delta > 0$ did not play a role in the proof of theorems 3.2 and 3.3 allowing us to disregard it.

Proposition 3.4. *Let μ be σ-ergodic, μ_λ is absolutely continuous for $\mathcal{L}$-a.e. $\lambda \in \left[2^{-h_\mu(\sigma)},\alpha_1\right]$.*

Proof. Apply theorem 3.3 for $E=\emptyset$. The proof in this case is identical to the one given by Peres and Solomyak in [8]. □

Proposition 3.5. *Let μ be σ-ergodic, μ_λ is absolutely continuous for $\mathcal{L}$-a.e. $\lambda \in \left[2^{-\frac{1}{3}\tilde{h}},\alpha_2\right]$ where $\tilde{h} = h_\mu\left(\sigma^3|\mathscr{A}_{\{i\in\mathbb{N}\,|\,i\equiv 0(mod3)\}}\right)$.*

Proof. Apply theorem 3.3 for $E = \{i\in\mathbb{N}\,|\,i\equiv 0\,(mod3)\}$. □

Proposition 3.6. *Let μ be σ-ergodic with $h = h_\mu(\sigma)$, μ_λ is absolutely continuous for $\mathcal{L}$-a.e. λ in $\left[2^{-\left(h-\frac{N-1}{N}\right)},\alpha_1^{\frac{1}{N}}\right]$ and $\left[2^{-\left(h-\frac{3N-2}{3N}\right)},\alpha_2^{\frac{1}{N}}\right]$, for all such N rendering these intervals non-empty.*

Proof. Denote $E_1^N = \{i\in\mathbb{N}\,|\,i\ne 0\,(modN)\}$ and $E_2^N = \{i\in\mathbb{N}\,|\,i\ne N,2N\,(mod3N)\}$. The fact that:

$$\mathcal{F}_{E_1^N} = \left\{\phi\left(x^N\right)\,|\,\phi\in\mathcal{F}\right\}$$

$$\mathcal{F}_{E_2^N} = \left\{\phi\left(x^N\right)\,|\,\phi\in\mathcal{F}_{\{i\in\mathbb{N}\,|\,i\equiv 0(mod3)\}}\right\}$$

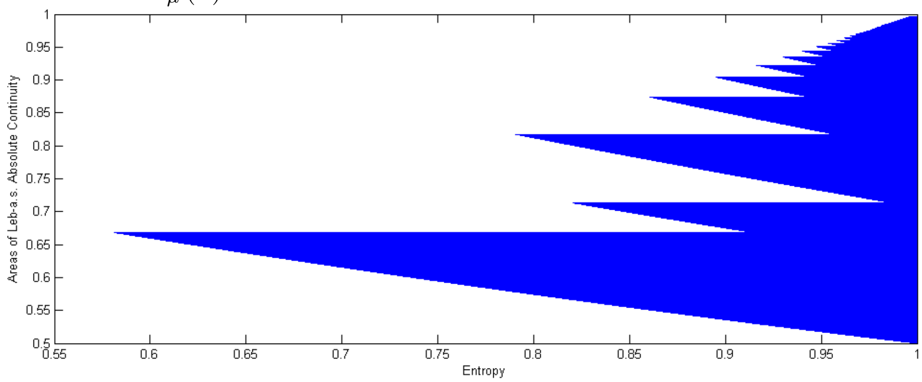
means $\left[\frac{1}{2},\alpha_1^{\frac{1}{N}}\right]$ is an interval of δ-transversality for $\mathcal{F}_{E_1^N}$ and $\left[\frac{1}{2},\alpha_2^{\frac{1}{N}}\right]$ is an interval of δ-transversality for $\mathcal{F}_{E_2^N}$. Using the Abramov-Rokhlin formula we can give crude lower bounds, depending only on h, for the corresponding conditional entropies:

$$h_\mu\left(\sigma^N|\mathscr{A}_{E_1^N}\right) \ge h_\mu\left(\sigma^N\right) - (N-1) = N\cdot h - (N-1)$$

$$h_\mu\left(\sigma^{3N}|\mathscr{A}_{E_2^N}\right) \ge 3N\cdot h - (3N-2)$$

Using theorem 3.3 implies the claim. □

Below is a graph depicting the areas of almost-sure absolute continuity assured for each value of $h_\mu(\sigma)$:



Corollary 3.7. *Given μ σ-ergodic with $h_\mu(\sigma) \geq 0.986916$ the measure μ_λ is absolutely continuous for $\mathcal{L}$-a.e. $\lambda \in \left[2^{-h_\mu(\sigma)}, \alpha_1^{\frac{1}{M}}\right]$ where:*

$$M = \max\left\{3 \leq N \in \mathbb{N} \,|\, h \geq 1 - \frac{1}{N} + \frac{-\log\alpha_1}{N-1}\right\}$$

.

Proof. The Intervals $\left[2^{-h}, \alpha_1\right]$ and $\left[2^{-\left(h-\frac{1}{3}\right)}, \alpha_2\right]$ intersect when $h \geq -\log\alpha_1 + \frac{1}{3}$, this is maintained since $-\log\alpha_1 + \frac{1}{3} \leq 0.915 \leq h$.
The intervals $\left[2^{-\left(h-\frac{1}{3}\right)}, \alpha_2\right]$ and $\left[2^{-\left(h-\frac{1}{2}\right)}, \alpha_1^{\frac{1}{2}}\right]$ intersect when $h \geq -\log\alpha_2 + \frac{1}{2}$, this is also maintained since $-\log\alpha_2 + \frac{1}{2} \leq 0.9869156 \leq h$.
The interval $\left[2^{-\left(h-\frac{N-1}{N}\right)}, \alpha_1^{\frac{1}{N}}\right]$ is non-empty whenever $h \geq 1 - \frac{1}{N} + \frac{-\log\alpha_1}{N}$.
The intervals $\left[2^{-\left(h-\frac{N-2}{N-1}\right)}, \alpha_1^{\frac{1}{N-1}}\right]$ and $\left[2^{-\left(h-\frac{N-1}{N}\right)}, \alpha_1^{\frac{1}{N}}\right]$ intersect whenever $h \geq 1 - \frac{1}{N} + \frac{-\log\alpha_1}{N-1}$. Assuming $3 \leq N \leq M$ assures[8]:

$$h \geq 1 - \frac{1}{N} + \frac{-\log\alpha_1}{N-1}$$

and consequently:

$$h \geq 1 - \frac{1}{N} + \frac{-\log\alpha_1}{N-1} > 1 - \frac{1}{N} + \frac{-\log\alpha_1}{N}$$

rendering the intersecting intervals non-empty. □

Example. Given a σ-ergodic measure μ with entropy $h_\mu(\sigma) > 0.99$, its projection μ_λ will be absolutely continuous for $\mathcal{L}$-a.e. $\lambda \in \left(2^{-h_\mu(\sigma)}, 0.98998\right)$.

Assuming lower bounds on the elements of the sequence $h_\mu\left(\sigma^N | \mathscr{A}_{E_1^N}\right)$ can yield stronger claims:

[8]Notice that for $N = 2$ this amount to $h > 1$ which is impossible.

Proposition 3.8. *Given μ σ-ergodic with $h_\mu(\sigma) > 0.986916$ and:*

$$h_\mu\left(\sigma^N|\mathscr{A}_{E_1^N}\right) \geq -\frac{N}{N-1}\log\alpha_1$$

for all $N \geq 3$, the measure μ_λ is absolutely continuous for $\mathcal{L}$-a.e. $\lambda \in \left(2^{-h_\mu(\sigma)}, 1\right)$.

Proof. The proof is the same as in 3.7 with condition $h_\mu\left(\sigma^N|\mathscr{A}_{E_1^N}\right) \geq -\frac{N}{N-1}\log\alpha_1$ assuring the intervals $\left[2^{-\frac{1}{N-1}h_\mu\left(\sigma^{N-1}|\mathscr{A}_{E_1^{N-1}}\right)}, \alpha_1^{\frac{1}{N-1}}\right]$ and $\left[2^{-\frac{1}{N}h_\mu\left(\sigma^N|\mathscr{A}_{E_1^N}\right)}, \alpha_1^{\frac{1}{N}}\right]$ are non-trivial and intersect for all $N \geq 3$. ☐

3.III. **Markov Measures.** Denote by $\mu_{p,P}$ the Markov measure with marginal $P = (p_{ij})_{i,j\in\{\pm1\}}$ and initial probability vector $p = \begin{pmatrix} p_1 \\ p_{-1}\end{pmatrix}$. We know:

$$h_{\mu^{p,P}}(\sigma) = -\sum_i p_i \sum_{i,j} p_{ij}\log p_{ij}$$

Denote by $\psi_N : \left(\{\pm1\}^{\mathbb{N}}, \sigma^N\right) \to \left(\left(\{\pm1\}^{N-1}\right)^{\mathbb{N}}, \sigma\right)$ the factor map projecting $\{\pm1\}^{\mathbb{N}}$ onto the $E_1^N = \{i \in \mathbb{N} \,|\, i \neq 0\,(modN)\}$ vector coordinates:

$$\psi_N\left((i_0, i_1, ..., i_{N-1}, i_N, ..., i_{,2N-1}, ...)\right) = \left(\begin{pmatrix} i_1 \\ \vdots \\ i_{N-1}\end{pmatrix}, \begin{pmatrix} i_{N+1} \\ \vdots \\ i_{2N-1}\end{pmatrix}, ...\right)$$

The projected measure $\psi_N\mu^p$ is itself a Markov measure with initial probability vector $\underline{q} = \left(p_{i_1}p_{i_1i_2}\cdots p_{i_{N-2}i_{N-1}}\right)_{\underline{i}\in\{\pm1\}^{N-1}}$ and marginal matrix:

$$P = \left(\left[\sum_{k=\pm1} p_{i_{N-1}k}p_{kj_1}\right]\cdot p_{j_1j_2}\cdots p_{j_{N-2}j_{N-1}}\right)_{\underline{i},\underline{j}\in\{\pm1\}^{N-1}}$$

We will Calculate the entropy of the factor, $h_{\psi_N\mu_{p,P}}(\sigma)$:

$$-\sum_{\underline{i}\in\{\pm1\}^{N-1}} p_{i_1}p_{i_1i_2}\cdots p_{i_{N-2}i_{N-1}} \sum_{\underline{j}\in\{\pm1\}^{N-1}} \left[\sum_k p_{i_{N-1}k}p_{kj_1}\right]\cdot p_{j_1j_2}\cdots$$

$$\cdots p_{j_{N-2}j_{N-1}}\log\left(\left[\sum_k p_{i_{N-1}k}p_{kj_1}\right]\cdot p_{j_1j_2}\cdots p_{j_{N-2}j_{N-1}}\right) =$$

$$= -\sum_{i_{N-1}} p_{i_{N-1}}\sum_{j_1}\left(\sum_k p_{i_{N-1}k}p_{kj_1}\right)\log\left(\sum_k p_{i_{N-1}k}p_{kj_1}\right)$$

$$-\sum_{k=1}^{N-2}\sum_{j_k} p_{j_k}\sum_{j_{k+1}} p_{j_kj_{k+1}}\log p_{j_kj_{k+1}} =$$

$$= h_{\mu^{p,P^2}}(\sigma) + (N-2)\cdot h_{\mu^{p,P}}(\sigma)$$

Using the Abramov-Rokhlin formula we receive:

$$\begin{aligned} h_{\mu^{p,P}}\left(\sigma^N|\mathscr{A}_{E_1^N}\right) &= h_{\mu^{p,P}}\left(\sigma^N\right) - h_{\psi_N\mu_{p,P}}\left(\sigma\right) = \\ &= 2\cdot h_{\mu^{p,P}}\left(\sigma\right) - h_{\mu^{p,P^2}}\left(\sigma\right) \end{aligned}$$

Notice the value is independent of N.
In addition:

$$h_{\mu^{p,P}}\left(\sigma^3|\mathscr{A}_{E_2^1}\right) = 3\cdot h_{\mu^{p,P}}\left(\sigma\right) - h_{\mu^{p,P^3}}\left(\sigma\right)$$

Using this we can state the following claim:

Proposition 3.9. *Given a σ-ergodic Markov measure $\mu^{p,P}$ with:*

$$3\cdot h_{\mu^{p,P}}\left(\sigma\right) - h_{\mu^{p,P^3}}\left(\sigma\right) \geq -3\log\alpha_1 \approx 1.7431634$$

$$2\cdot h_{\mu^{p,P}}\left(\sigma\right) - h_{\mu^{p,P^2}}\left(\sigma\right) \geq -2\log\alpha_2 \approx 0.9738312$$

its projection $\mu_\lambda^{p,P}$ is absolutely continuous for $\mathcal{L}$-a.e. $\lambda \in \left(2^{-h_{\mu^{p,P}}(\sigma)}, 1\right)$.

Proof. Denote $C = h_{\mu^{p,P}}\left(\sigma^N|\mathscr{A}_{E_1^N}\right)$. The Intervals $\left[2^{-h(\sigma)}, \alpha_1\right]$ and $\left[2^{-\frac{1}{3}h_{\mu^p}\left(\sigma^3|\mathscr{A}_{E_2^1}\right)}, \alpha_2\right]$ intersect when $h_{\mu^p}\left(\sigma^3|\mathscr{A}_{E_2^1}\right) \geq -3\log\alpha_1$.
The intervals $\left[2^{-\frac{1}{3}h_{\mu^p}\left(\sigma^3|\mathscr{A}_{E_2^1}\right)}, \alpha_2\right]$ and $\left[2^{-\frac{1}{2}C}, \alpha_1^{\frac{1}{2}}\right]$ intersect when $C \geq -2\log\alpha_2$.
The interval $\left[2^{-\frac{1}{N}C}, \alpha_1^{\frac{1}{N}}\right]$ is non-empty whenever $C \geq -\log\alpha_1$.
For all $N \geq 3$, the intervals $\left[2^{-\frac{C}{N-1}}, \alpha_1^{\frac{1}{N-1}}\right]$ and $\left[2^{-\frac{C}{N}}, \alpha_1^{\frac{1}{N}}\right]$ intersect whenever $C \geq -\frac{3}{2}\log\alpha_1$. Since $-2\log\alpha_2 \geq -\frac{3}{2}\log\alpha_1$ all these conditions are satisfied. □

Denote by μ^p the Markov measure changing signs with probability $1-p$ and leaving signs unchanged with probability p, i.e. the Markov measure with marginal $\begin{pmatrix} p & 1-p \\ 1-p & p \end{pmatrix}$.

Corollary 3.10. *The projection μ_λ^p is absolutely continuous for $\mathcal{L}$-a.e. $\lambda \in \left(2^{-H(p,1-p)}, 1\right)$, given any $p \in [0.432455, 0.567545]$*

Proof. In this case:

$$C = h_{\mu^p}\left(\sigma^N|\mathscr{A}_{E_1^N}\right) = 2H\left(p, 1-p\right) - H\left(p^2 + (1-p)^2, 2p\left(1-p\right)\right)$$

$$h_{\mu^p}\left(\sigma^3|\mathscr{A}_{E_2^1}\right) = 3\cdot H\left(p, 1-p\right) - H\left(p^3 + 3p\left(1-p\right)^2, \left(1-p\right)^3 + 3\left(1-p\right)p^2\right)$$

Calculation shows:

$$h_{\mu^p}\left(\sigma^3|\mathscr{A}_{E_2^1}\right) \geq -3\log\alpha_1 \Longleftarrow p \in [0.329101, 0.670899]$$

$$C \geq -2\log\alpha_2 \Longleftarrow p \in [0.432455, 0.567545]$$

Hence the conditions of proposition 3.9 hold for all $p \in [0.432455, 0.567545]$, as stated. □

Another simple family of Markov Measures are the biased Bernoulli measures. The result received here is strictly weaker than the one by Peres and Solomyak in [8].

Corollary 3.11. *For the biased Bernoulli convolution ν^p with $p \in [0.405058, 0.594942]$, the projection ν^p_λ is absolutely continuous for $\mathcal{L}$-a.e. $\lambda \in \left(2^{-H(p,1-p)}, 1\right)$.*

Proof. Denote $h = h_{\nu^p}(\sigma) = H(p, 1-p)$ and notice that $h\left(\sigma^3 | \mathscr{A}_{E_2^1}\right) = 2h$ and for all N, $h_\mu\left(\sigma^N | \mathscr{A}_{E_1^N}\right) = h$. The assumption assures $h > 0.973832$, implying the conditions of proposition 3.9 are satisfied. □

4. Exceptional Set is G_δ

Lemma 4.1. *Let μ be a non-atomic σ-invariant measure on Ω, if μ_λ has an atom then λ is a root of a polynomial with coefficients in $\{\pm 1, 0\}$.*

Proof. Denote:

$$A_0^\lambda = \{(\omega, \tau) \mid \pi_\lambda(\omega) = \pi_\lambda(\tau)\} \subseteq \Omega \times \Omega$$

Assuming μ_λ has an atom assures $\mu \times \mu\left(A_0^\lambda\right) > 0$. The measure $\mu \times \mu$ is $\sigma \times \sigma$-invariant and hence for $\mu \times \mu$-a.e. $(\omega, \tau) \in A_0^\lambda$ there exists a sequence $n_k \to \infty$ for which $\forall k \quad (\sigma \times \sigma)^{n_k}(x) \in A_0^\lambda$. This means that:

$$\pi_\lambda(\omega) - \pi_\lambda(\tau) = \pi_\lambda(\sigma^{n_k}\omega) - \pi_\lambda(\sigma^{n_k}\tau) = 0$$

Denoting $a_l = \frac{1}{2}(\omega_l - \tau_l)$ we receive:

$$0 = \sum_{l=0}^{\infty} a_l \lambda^l = \sum_{l=0}^{n_k - 1} a_l \lambda^l + \lambda^{n_k} \overbrace{\left(\sum_{l=n_k}^{\infty} a_l \lambda^{l-n_k}\right)}^{=0} = \sum_{l=0}^{n_k-1} a_l \lambda^l$$

This being true for all $n_k \to \infty$ leaves two options: either λ is a root of a polynomial with coefficients in $\{\pm 1, 0\}$ or $\forall l \in \mathbb{N} \;\; a_l = 0$.
The latter cannot hold for $\mu \times \mu$-a.e. $(\omega, \tau) \in A_0^\lambda$ since that would mean:

$$A_0^\lambda \underset{\mu\times\mu}{\subseteq} \Delta = \{(\omega, \omega) \mid \omega \in \Omega\}$$

whereas $\mu \times \mu(\Delta) = 0$ by Fubini and the non-atomicity of μ. □

Denote $\mathcal{A}_{\{\pm 1, 0\}} = \{x \in \mathbb{R} \mid x \text{ is a root of a polynomial with coefficients in } \{\pm 1, 0\}\}$. The rest of the proof goes along the lines of proposition 8.1 in [6]:

Proposition 4.2. *Given an interval (a, b) the function $\lambda \mapsto \mu_\lambda(a, b)$ is continuous on $\left(\frac{1}{2}, 1\right) \backslash \mathcal{A}_{\{\pm 1, 0\}}$.*

Proof. For $a < b$:

$$\mu_\lambda(a, b) = \int \chi_{\{\eta \mid \pi_\lambda(\eta) \in (a,b)\}} d\mu$$

Given a sequence $\lambda_n \to \lambda$ where $\lambda \in \left(\frac{1}{2}, 1\right) \backslash \mathcal{A}_{\{\pm 1, 0\}}$, we need to show $\mu_{\lambda_n}(a, b) \to \mu_\lambda(a, b)$ and in order to do so we will use the dominated convergence theorem. All we need to show is that the functions:

$$f_n = \chi_{\{\eta \mid \pi_{\lambda_n}(\eta) \in (a,b)\}} : \Omega \to \mathbb{R}$$

converge pointwise μ-a.e. to:

$$f = \chi_{\{\eta \,|\, \pi_\lambda(\eta) \in (a,b)\}}$$

Fix an $\omega \in \Omega$ and denote the function $\varphi_\omega(\lambda) = \pi_\lambda(\omega)$. This Function is continuous and thus if $\varphi_\omega(\lambda) \in (a,b)$ then there exists an N for which $\forall n > N \;\; \varphi_\omega(\lambda_n) \in (a,b)$ meaning $f_n(\omega) \equiv 1$ for all $n > N$ and evidently $f_n(\omega) \to f(\omega)$.
The same argument holds when $\varphi_\omega(\lambda) \in int(\mathbb{R} \setminus (a,b))$. The case where $\varphi_\omega(\lambda) = \pi_\lambda(\omega) \in \partial(a,b) = \{a,b\}$ can be avoided since $\mu(\{\omega \,|\, \varphi_\omega(\lambda) \in \{a,b\}\}) = 0$ as a consequence of lemma 4.1 and the assumption $\lambda \notin \mathcal{A}_{\{\pm1,0\}}$. □

Corollary 4.3. $S_\perp^\mu = \left\{ \lambda \in \left(\frac{1}{2}, 1\right) \,\middle|\, \mu_\lambda \text{ is singular} \right\}$ *is a* G_δ *set.*

Proof. Denote $X = \left(\frac{1}{2}, 1\right) \setminus \mathcal{A}_{\{\pm1,0\}}$. If we show $S_\perp^\mu \cap X$ is G_δ with respect to the induced metric on X we will conclude $S_\perp^\mu$ is G_δ, since if:

$$S_\perp^\mu \cap X = \bigcap_i (U_i \cap X)$$

where U_i are open in $\mathbb{R}$, then:

$$S_\perp^\mu = \left(\bigcap_i U_i \right) \bigcap \left(\bigcap_{\alpha \in \mathcal{A}_{\{\pm1,0\}} \setminus S_\perp} \left(\left(\frac{1}{2}, \alpha \right) \cup (\alpha, 1) \right) \right)$$

as required (recall $\mathcal{A}_{\{\pm1,0\}}$ is countable).
Let $\mathcal{G}$ be the collection of all finite unions of open intervals $(a,b) \subseteq \mathbb{R}$. By proposition 4.2, for any $G \in \mathcal{G}$ the set $\left\{\lambda \in X \,|\, \mu_\lambda(G) > \frac{1}{2}\right\}$ is open in X and thus:

$$\bigcap_n \bigcup_{\substack{G \in \mathcal{G} \\ \mathcal{L}(G) < 2^{-n}}} \left\{ \lambda \in X \,|\, \mu_\lambda(G) > \frac{1}{2} \right\}$$

is a G_δ set in X. We will prove:

$$S_\perp^\mu \cap X = \bigcap_n \bigcup_{\mathcal{L}(G) < 2^{-n}} \left\{ \lambda \in X \,|\, \mu_\lambda(G) > \frac{1}{2} \right\} \tag{4.1}$$

Let $\lambda \in S_\perp^\mu$, there exists a set $A \subseteq \mathbb{R}$ with $\mu_\lambda(A) = 1$ and $\mathcal{L}(A) = 0$. Due to the properties of Lebesgue measure, for any n there exists a cover by open intervals $\{U_i\}_{i \in \mathbb{N}}$ of A with $\sum_{i=1}^\infty |U_i| < 2^{-n}$. On the other hand, there exists a k for which $\sum_{i=1}^k \mu_\lambda(U_i) > \frac{1}{2}$ giving us the required set $G = \cup_{i=1}^k U_i \in \mathcal{G}$.
Now assume λ is an element of the right hand side of (4.1). For every n denote $G_n \in \mathcal{G}$ to be a finite union of open intervals with $\mathcal{L}(G_n) < 2^{-n}$ and $\mu_\lambda(G_n) > \frac{1}{2}$. Let A be the limit superior of the sequence G_n:

$$A = \bigcap_n \bigcup_{k=n}^\infty G_n$$

On the one hand, for every $n \in \mathbb{N} \;\; A \subseteq \bigcup_{k=n}^\infty G_n$ giving:

$$\mathcal{L}(A) \le \mathcal{L}\left(\bigcup_{k=n}^\infty G_n \right) \le \sum_{k=n}^\infty \mathcal{L}(G_n) = 2^{-n+1}$$

and consequently $\mathcal{L}(A)=0$. On the other hand, by the reverse Fatou lemma:

$$\mu_\lambda(A)=\int \overline{\lim}_{n\to\infty}\chi_{G_n}d\mu_\lambda \geq \overline{\lim}_{n\to\infty}\int \chi_{G_n}d\mu_\lambda = \overline{\lim}_{n\to\infty}\mu_\lambda(G_n)\geq \frac{1}{2}$$

Using the fact that μ_λ is of pure type assures $\lambda \in S_\perp^\mu$, as required. □

5. Establishing Transversality

5.I. **Simple Proof of Transversality.** This proof is identical to the one in [7] with slight changes of notation and terminology. Recall our definition of δ-transversality:

Definition 5.1. Given a set of indices $E\subseteq\mathbb{N}$, we say the interval $I\subseteq\left[\frac{1}{2},1\right]$ is an interval of *δ-transversality for the family of functions $\mathcal{F}_E$* if for any sub-interval $I_0=[\lambda_0,\lambda_1]\subseteq I$ and all $\phi\in\mathcal{F}_E$ and $r>0$:

$$\mathcal{L}\left\{x\in I_0 \mid |\phi(x)|\leq r\right\}\leq 2\delta^{-1}\lambda_0^{-\wedge(\phi)}r$$

where $\wedge\left(\sum_{k=0}^{\infty}a_kx^k\right)=\inf_k(a_k\neq 0)\in\mathbb{N}\cup\{\infty\}$.

Notice that any $\phi\in\mathcal{F}_E$ can be presented as:

$$\phi(x)=\pm x^s\cdot g(x)$$

where $s=\wedge(\phi)$ and g is of the form:

$$g(x)=1+\sum_{\substack{i\in(E^c-s-1)\\ 0\leq i}}a_ix^i$$

where $a_i\in\{\pm1,0\}$ and $E^c=\mathbb{N}\backslash E$. In this case we say that g *is of the form* (E,s). When E is N_0-periodic there are a finite number of such (E,s)-forms corresponding to different $s\in\{e\in E^c \mid e<N_0\}$.

Notice that proving that for all $s\in\{e\in E^c\mid e<N_0\}$, all functions of (E,s)-form satisfy:

$$\mathcal{L}\left\{x\in I \mid |g(x)|\leq r\right\}\leq 2\delta^{-1}r \tag{5.1}$$

for some $\delta>0$ and any $r>0$, implies δ-transversality for the whole family $\mathcal{F}_E$ on I, since:

$$\begin{aligned}\mathcal{L}\left\{x\in I_0\mid|\phi(x)|\leq r\right\} &\leq \mathcal{L}\left\{x\in I_0\mid |g(x)|\leq\lambda_0^{-s}r\right\}\leq\\ &\leq \mathcal{L}\left\{x\in I\mid |g(x)|\leq\lambda_0^{-s}r\right\}\leq 2\delta^{-1}\lambda_0^{-s}r\end{aligned}$$

for any $I_0\subseteq I$.

Definition 5.2. Let g be a function of (E,s)-form satisfying for all $x\in I$:

$$g(x)<\delta \implies g'(x)<-\delta$$

Then we say g satisfies the δ-transversality condition on I.

Remark. This is the original δ-transversality condition as defined in [7].

Lemma 5.3. *Let g be a function of (E,s)-form satisfying the δ-transversality condition on I, then g satisfies condition 5.1 for all $r>0$.*

Proof. When $r \geq \delta$ the claim is trivially true since:

$$\mathcal{L}\left\{\lambda \in I \mid |g(x)| \leq r\right\} \leq |I| < 2$$

Given some $r < \delta$, by assumption whenever $|g(x)| \leq r$ it is monotone decreasing with slope$< -\delta$ so the function g intersects $[-r, r]$ at most once and for time$\leq 2\delta^{-1}r$ as required. ☐

Definition 5.4. A power series $h(x)$ is called an $(E,s)-(*)$-*function* if for some $k \geq 1$ and $a_k \in [-1,1]$:

$$h(x) = 1 - \left(\sum_{\substack{i\in(E^c-s-1)\\0\leq i\leq k-1}} x^i\right) + a_k x^k + \left(\sum_{\substack{i\in(E^c-s-1)\\k+1\leq i}} x^i\right)$$

Lemma 5.5. *Suppose that an $(E,s)-(*)$-function h satisfies:*

$$h(x_0) > \delta \quad and \quad h'(x_0) < -\delta$$

for some $x_0 \in \left[\frac{1}{2}, 1\right]$ and $\delta \in (0,1)$. Then all functions of (E,s)-form satisfy the δ-transversality condition on $\left[\frac{1}{2}, x_0\right]$.

Proof. $h'(0) < -\delta$, since either $h'(0) = -1$ when $k > 1$ or:

$$h'(0) = a_1 \leq a_1 + \sum_{\substack{i\in(E^c-s-1)\\2\leq i}} i x^i = h'(x_0) < -\delta$$

when $k = 1$. Since $\lim_{x\to 1} h'(x) = \infty$, assuming $h'(x) \geq -\delta$ for some $x \in [0, x_0]$ would imply h'' has at least two zeros in $[0,1)$ contradicting the fact that h'' is a power series with at most one coefficient sign change. Hence $h'(x) < -\delta$ for all $x \in [0, x_0]$. Adding the fact that $h(0) = 1$ implies $h(x) > \delta$ for all $x \in [0, x_0]$.

Let g be a power series of form (E,s) and denote $f = g - h$. By definition f is of the form:

$$f(x) = \sum_{\substack{i\in(E^c-s-1)\\1\leq i\leq l}} c_i x^i - \sum_{\substack{i\in(E^c-s-1)\\l+1\leq i}} c_i x^i$$

where $l \in \{k-1, k\}$ and all $c_i \geq 0$. Hence for all $x \in [0, x_0]$:

$$g(x) < \delta \;\Rightarrow\; f(x) < 0 \;\Rightarrow\; f'(x) < 0 \;\Rightarrow\; g'(x) < -\delta$$

Where the middle implication is a consequence of one coefficient sign change:

$$\begin{aligned} f(x) \quad < \quad 0 &\Rightarrow \left(\sum_{\substack{i\in(E^c-s-1)\\1\leq i\leq l}} c_i x^i\right) < \left(\sum_{\substack{i\in(E^c-s-1)\\l+1\leq i}} c_i x^i\right) \\ &\Rightarrow \left(\sum_{\substack{i\in(E^c-s-1)\\1\leq i\leq l}} c_i i x^{i-1}\right) < \left(\sum_{\substack{i\in(E^c-s-1)\\l+1\leq i}} c_i i x^{i-1}\right) \\ &\Rightarrow f'(x) < 0 \end{aligned}$$

☐

So we have reduced the problem of proving δ-transversality for the whole family $\mathcal{F}_E$ to a problem of finding a finite number $(|\{e \in E^c \mid e < N_0\}|)$ of $(*)$-functions satisfying the conditions of lemma 5.5.

Proposition 5.6.

(1) $\left[\frac{1}{2}, 0.639774\right]$ *is an interval of δ-transversality for the family $\mathcal{F}$.*
(2) $\left[\frac{1}{2}, \alpha_2\right]$ *is an interval of δ-transversality for the family $\mathcal{F}_{\{i\in\mathbb{N}\,|\,i\equiv 0(mod3)\}}$, with $\alpha_2 = 0.713549$.*

Proof.

(1) When $E = \emptyset$ we only need to check one form of $(\emptyset, 0)$-functions. The $(*)$-function:

$$h_0(x) = 1 - x - x^2 - x^3 + 0.08x^4 + \sum_{i=5}^{\infty} x^i$$

satisfies:

$$h_0(0.639774) > 2 \cdot 10^{-7}$$
$$h_0'(0.639774) < -0.2 < -2 \cdot 10^{-7}$$

proving $\left[\frac{1}{2}, 0.639774\right]$ is an interval of $2 \cdot 10^{-7}$-transversality for $\mathcal{F}$.

(2) For $E = \{i \in \mathbb{N} \mid i \equiv 0\,(mod 3)\}$ we need to address two forms
   (a) $(E, 1)$-form - The $(*)$-function:

$$h_1(x) = 1 - x - x^3 - x^4 + 0.855x^6 + x^7 + \frac{x^9 + x^{10}}{1 - x^3}$$

   satisfies:

$$h_1(\alpha_2) > 2 \cdot 10^{-9}$$
$$h_1'(\alpha_2) < -0.3 < -2 \cdot 10^{-9}$$

   (b) $(E, 2)$-form - The $(*)$-function:

$$h_2(x) = 1 - x^2 - x^3 - x^5 - 0.5x^6 + \frac{x^8 + x^9}{1 - x^3}$$

   satisfies:

$$h_2(\alpha_2) > 0.05$$
$$h_2'(\alpha_2) < -2 < -0.05$$

   Assuring $\left[\frac{1}{2}, \alpha_2\right]$ is an interval of $2 \cdot 10^{-9}$-transversality for the family $\mathcal{F}_E$.

□

5.II. **Estimation of Double Zeros.** In [10] Pablo Shmerkin and Boris Solomyak extended the interval of transversality for the family $\mathcal{F}$ significantly by addressing the equivalent problem of estimating the minimal double zero attained by a power series of $(\emptyset, 0)$-form:

$$1 + \sum_{n=1}^{\infty} a_n x^n \quad a_{n\in\{\pm 1, 0\}} \tag{5.2}$$

Denote:

$$X_2 = \{x \in (0,1) \mid \exists f \text{ of } (\emptyset, 0)\text{-form with } f(x) = f'(x) = 0\}$$

The following result was established:

Theorem 5.7.

$$\alpha = \min X_2 \in (0.6684755, 0.6684757)$$

We will not elaborate on the proof of this theorem but rather prove its relevance to δ-transversality, as done in [11]:

Lemma 5.8. *For all $\varepsilon > 0$ there exists a $\delta > 0$ such that any function g of $(\emptyset, 0)$-form satisfies:*

$$\forall x \in [0, \alpha - \varepsilon] \quad |g(x)| < \delta \;\Rightarrow\; |g'(x)| < -\delta$$

Proof. Assume in contradiction an existence of a sequence of numbers $x_i \in [0, \alpha - \varepsilon]$ and functions g_i of $(\emptyset, 0)$-form satisfying $g_i(x_i) \underset{i\to\infty}{\longrightarrow} 0$ and $g_i'(x_i) \underset{i\to\infty}{\longrightarrow} 0$. WLOG we may assume $x_i \underset{i\to\infty}{\longrightarrow} x \in [0, \alpha - \varepsilon]$ and $g_i \underset{i\to\infty}{\longrightarrow} g$ coefficient by coefficient. The function g is also of $(\emptyset, 0)$-form, but this would mean:

$$g(x) = \lim_{i\to 0} g_i(x_i) = 0$$

$$g'(x) = \lim_{i\to 0} g_i'(x_i) = 0$$

where $x < \alpha$ in contradiction to the definition of α. □

In particular, there exists a δ for which all g of $(\emptyset, 0)$-form receiving values in $(-\delta, \delta)$ somewhere along the interval $\left[\frac{1}{2}, \alpha_1\right]$, for $\alpha_1 = 0.668475$, do so with slope greater than δ (in absolute value). As shown in lemma 5.3, this implies that for any $r < \delta < 1$, each interval in $g^{-1}((-r, r))$ is of length$\leq 2\delta^{-1} r$.
In order to deduce a condition of the sort:

$$\mathcal{L}\left\{x \in \left[\frac{1}{2}, \alpha_1\right] \mid |g(x)| \leq r\right\} \leq C\delta^{-1} r$$

one only needs to give a bound on the number of intervals in $g^{-1}((-\delta, \delta))$:

Lemma 5.9. *There exists a constant C such that for all g of $(\emptyset, 0)$-form, $U = g^{-1}((-\delta, \delta))$ is a union of at most C intervals.*

Proof. For all $x \in (0, 1)$:

$$|g'(x)| \leq \sum_{i=0}^{\infty} i x^{i-1} = \frac{1}{(1-x)^2} \leq \frac{1}{(1-\alpha_1)^2}$$

Every interval $J \subseteq U$ admits $g(J) = (-\delta, \delta)$, since g is monotone on J and $g(0) = 1 > \delta$. Therefore $|J| \geq 2\delta(1-\alpha_1)^2$. The fact that $\mathcal{L}(U) \leq 1$ assures the required constant is $C = \lfloor \frac{1}{2\delta(1-\alpha_1)^2} \rfloor$. □

Corollary 5.10. $\left[\frac{1}{2}, \alpha_1\right]$ *is an interval of $C^{-1}\delta$-transversality for the family $\mathcal{F}$.*

6. Appendix

6.I. Full Proof of Pure Type Using the Density Function.

We denote the α-dimensional Hausdorff measure on $\mathbb{R}$ by $\mathcal{H}^{\alpha}$. We prove the following:

Proposition 6.1. *For any σ-ergodic measure μ and any $\lambda \in (0,1)$, the measure μ_{λ} is of pure type with respect to $\mathcal{H}^{\alpha}$, i.e. either $\mu_{\lambda} \ll \mathcal{H}^{\alpha}$ or $\mu_{\lambda} \perp \mathcal{H}^{\alpha}$.*

Remark. We interpret $\mu_{\lambda} \ll \mathcal{H}^{\alpha}$ as the property that for any set $E \subseteq \mathbb{R}$:

$$\mathcal{H}^{\alpha}(E) = 0 \Rightarrow \mu_{\lambda}(E) = 0$$

and $\mu_{\lambda} \perp \mathcal{H}^{\alpha}$ as the existence of a set $E' \subseteq \mathbb{R}$ for which $\mu_{\lambda}(\mathbb{R} \backslash E') = 0$ and $\mathcal{H}^{\alpha}(E') = 0$.

Definition. *The upper α-dimensional density of a measure ν at $x \in \mathbb{R}^d$ is:*

$$D_{\alpha}^{+}(\nu, x) = \limsup_{r \to 0} \frac{\nu(B_r(x))}{(2r)^{\alpha}}$$

where $B_r(x)$ is the closed ball of radius r around x.

Denote:

$$A_{\infty}^{\alpha} = \left\{ \omega \in \Omega \,\middle|\, D_{\alpha}^{+}(\mu_{\lambda}, \pi_{\lambda}(\omega)) < \infty \right\} = \pi_{\lambda}^{-1} \circ \left(D_{\alpha}^{+}\right)^{-1}([0, \infty))$$

Lemma 6.2. *$D_{\alpha}^{+}(\mu, \cdot)$ is measurable.*

Proof. First we notice that $\frac{\mu(B_r(x))}{(2r)^{\alpha}}$ is right-continuous with respect to r, since μ is finite and $\lim_{s \searrow r} B_s(x) = B_r(x)$. This allows us to restrict the limit to $r \searrow 0$ along the rationals. Second, for each $r > 0$ the function $D_{\alpha}^{r}(x) = \frac{\mu(B_r(x))}{(2r)^{\alpha}}$ is upper semi-continuous with respect to x and thus measurable. This is seen by noticing that for any $s > r$ and $x_n \to x$ we have $B_r(x_n) \subseteq B_s(x)$ for large enough n, which implies $D_{\alpha}^{r}(x_n) \leq \frac{\mu(B_s(x))}{(2r)^{\alpha}}$. But since $\mu(B_s(x))$ is right-continuous with respect to s, taking $s \searrow r$ and a $\limsup_{n \to \infty}$ gives:

$$\limsup_{n \to \infty} D_{\alpha}^{r}(x_n) \leq D_{\alpha}^{r}(x)$$

as required. Hence we see that D_{α}^{+} is a limsup of a sequence of measurable functions and thus is itself measurable.

∎

This show A_{∞}^{α} is measurable.

Proposition 6.3. *A_∞^α is σ-invariant in Ω up to a μ-null set.*

Proof. We will begin by pointing out a few identities.
Since $\mu = \sigma\mu$ and $\mu = \mu|_{[1]} + \mu|_{[-1]}$ we get $\mu = \sigma\mu|_{[1]} + \sigma\mu|_{[-1]}$ and thus:

$$\mu_\lambda = \pi_\lambda\mu = \pi_\lambda\sigma\mu|_{[1]} + \pi_\lambda\sigma\mu|_{[-1]} \tag{6.1}$$

We notice that for any measurable set $E \subseteq \mathbb{R}$ we have:

$$\begin{aligned}\pi_\lambda\sigma\mu|_{[1]}(E) &= \mu|_{[1]}\left(\sigma^{-1}\left(\pi_\lambda^{-1}E\right)\right) = \mu|_{[1]}\left(\sigma^{-1}\left(\pi_\lambda^{-1}E\right)\cap[1]\right) = \\ &= \mu|_{[1]}\left(\pi_\lambda^{-1}\left(\varphi_+E\right)\cap[1]\right) = \mu|_{[1]}\left(\pi_\lambda^{-1}\left(\varphi_+E\right)\right) = \pi_\lambda\mu|_{[1]}\left(\varphi_+E\right)\end{aligned} \tag{6.2}$$

where the crucial equality is given by the fact that for all $F \subseteq \Omega$:

$$\sigma^{-1}F \cap [1] = \left\{1\omega = (1\omega_0\omega_1...) \,\middle|\, \omega = (\omega_0\omega_1...) \in F\right\}$$

hence $\pi_\lambda\left(\sigma^{-1}F\cap[1]\right) = \varphi_+\left(\pi_\lambda F\right)$. A similar identity holds for $[-1]$. If we denote $\mu_\lambda^\pm = \pi_\lambda\mu|_{[\pm1]}$ we receive from 6.1+6.2 the following identity:

$$\mu_\lambda = \varphi_+^{-1}\mu_\lambda^+ + \varphi_-^{-1}\mu_\lambda^- \tag{6.3}$$

Remark. Notice this identity is not equivalent to self-similarity, i.e.σ-invariance does not imply self-similarity of the projected measure.

This shows that for μ-a.e. $\omega \in \Omega$ and every $r > 0$ we have:

$$\begin{aligned}\mu\left(B_r\left(\pi_\lambda\omega\right)\right) &= \varphi_+^{-1}\mu_\lambda^+\left(B_r\left(\pi_\lambda\omega\right)\right) + \varphi_-^{-1}\mu_\lambda^-\left(B_r\left(\pi_\lambda\omega\right)\right) = \\ &= \mu_\lambda^+\left(\varphi_+B_r\left(\pi_\lambda\omega\right)\right) + \mu_\lambda^-\left(\varphi_-B_r\left(\pi_\lambda\omega\right)\right) = \\ &= \mu_\lambda^+\left(B_{\lambda r}\left(\pi_\lambda(1\omega)\right)\right) + \mu_\lambda^-\left(B_{\lambda r}\left(\pi_\lambda(-1\omega)\right)\right)\end{aligned}$$

Dividing by $(2r)^\alpha$ we receive:

$$\frac{\mu\left(B_r\left(\pi_\lambda\omega\right)\right)}{(2r)^\alpha} \geq \frac{\mu_\lambda^\pm\left(B_{\lambda r}\left(\pi_\lambda(\pm1\omega)\right)\right)}{(2r)^\alpha} = \lambda^\alpha\frac{\mu_\lambda^\pm\left(B_{\lambda r}\left(\pi_\lambda(\pm1\omega)\right)\right)}{(2\lambda r)^\alpha}$$

Taking limsup-s gives:

$$D_\alpha^+\left(\mu,\pi_\lambda\omega\right) \geq \lambda^\alpha\cdot D_\alpha^+\left(\mu_\lambda^\pm,\pi_\lambda(\pm1\omega)\right) \tag{6.4}$$

Now since μ and $\mu_\lambda^\pm$ are all finite measures we can use Lebesgue's decomposition theorem to decompose $\mu_\lambda = \nu_{\pm ac} + \nu_{\pm s}$ with respect to $\mu_\lambda^\pm$. Since $\mu_\lambda^+ \perp \nu_{+s}$ and $\mu_\lambda^- \perp \nu_{-s}$ there exist two measurable sets C_+ and C_- for which $\mu_\lambda^\pm\left(\mathbb{R}\backslash C_\pm\right) = \nu_{\pm ac}\left(\mathbb{R}\backslash C_\pm\right) = 0$ and $\nu_{\pm s}\left(C_\pm\right) = 0$. This gives $\mu_\lambda|_{C_\pm} = \nu_{\pm ac}$. We also notice $\mathbb{R} \underset{\mu_\lambda}{=} C_+ \cup C_-$, otherwise there would be a set $E \subseteq \mathbb{R}\backslash\left(C_+\cup C_-\right)$ with $\mu_\lambda(E) > 0$ such that $\mu_\lambda|_E \perp \mu_\lambda^+$ and $\mu_\lambda|_E \perp \mu_\lambda^-$ in contradiction to the fact that $\mu_\lambda = \mu_\lambda^+ + \mu_\lambda^-$. Hence WLOG we may assume that $\mathbb{R} = C_+ \cup C_-$ (strict equality).
If we were to assume that $\mu_\lambda\left(C_-\right) = 0$ we would get $\mu_\lambda^-\left(C_-\right) = 0$ and consequently $\mu_\lambda^- \equiv 0$. But this would mean that $\mu([-1]) = 0$ and thus that $\mu([1]) = 1$ which would in turn imply $\mu\left(\sigma^{-1}[1]\cap[1]\right) = \mu([11]) = 1$ and so forth, meaning $\mu \equiv \delta_{11...}$.

In this case $D_{\alpha}^{+}\left(\pi_{\lambda}\delta_{11\ldots},\cdot\right)\equiv 0$ rendering the proposition trivial. The same happens when assuming $\mu_{\lambda}\left(C_{+}\right)=0$. Therefore we may assume from now on that both $C_{\pm}$ are of positive measure.

We will notice in addition that $[1]\underset{\mu}{\subseteq}\pi_{\lambda}^{-1}C_{+}$ and $[-1]\underset{\mu}{\subseteq}\pi_{\lambda}^{-1}C_{-}$, since if $\mu\left([1]\setminus\pi_{\lambda}^{-1}C_{+}\right)>0$ then $\mu|_{[1]}\left([1]\setminus\pi_{\lambda}^{-1}C_{+}\right)>0$ and by definition $\mu_{\lambda}^{+}\left(\pi_{\lambda}\left([1]\setminus\pi_{\lambda}^{-1}C_{+}\right)\right)>0$. But this would mean that $\mu_{\lambda}^{+}\left(\mathbb{R}\setminus C_{+}\right)>0$ in contradiction to the definition of C_{+}. Similarly for C_{-}.

We denote $f_{\pm}=\dfrac{dv_{\pm ac}}{d\mu_{\lambda}^{\pm}}$ the Radon derivatives and choose them to take values only in $[0,\infty)$.[9]

By Besicovitch's density theorem, since μ_{λ} is a probability measure on $\mathbb{R}$ and $\mu_{\lambda}\left(C_{\pm}\right)>0$ we have:

$$\lim_{r\to 0}\frac{\mu_{\lambda}\left(B_{r}\left(x\right)\cap C_{\pm}\right)}{\mu_{\lambda}\left(B_{r}\left(x\right)\right)}=1$$

for μ_{λ}-a.e. $x\in C_{\pm}$. Now since $\mu_{\lambda}|_{C_{\pm}}=\nu_{\pm ac}$ we can use the differentiation theorem (see 2.14 in [5]) to receive:

$$\lim_{r\to 0}\frac{\mu_{\lambda}\left(B_{r}\left(x\right)\cap C_{\pm}\right)}{\mu_{\lambda}^{\pm}\left(B_{r}\left(x\right)\right)}=f_{\pm}\left(x\right)$$

for μ_{λ}-a.e. $x\in C_{\pm}$.

This gives us for μ-a.e. $\omega\in\Omega$, $\pi_{\lambda}\left(\pm1\omega\right)\in C_{\pm}$ and:

$$\lim_{r\to 0}\frac{\mu_{\lambda}\left(B_{r}\left(\pi_{\lambda}\left(\pm1\omega\right)\right)\right)}{\mu_{\lambda}^{\pm}\left(B_{r}\left(\pi_{\lambda}\left(\pm1\omega\right)\right)\right)}=\lim_{r\to 0}\left(\frac{\mu_{\lambda}\left(B_{r}\left(\pi_{\lambda}\left(\pm1\omega\right)\right)\right)}{\mu_{\lambda}^{\pm}\left(B_{r}\left(\pi_{\lambda}\left(\pm1\omega\right)\right)\right)}\cdot\frac{\mu_{\lambda}\left(B_{r}\left(\pi_{\lambda}\left(\pm1\omega\right)\right)\cap C_{\pm}\right)}{\mu_{\lambda}\left(B_{r}\left(\pi_{\lambda}\left(\pm1\omega\right)\right)\right)}\right)=$$

$$=\lim_{r\to 0}\frac{\mu_{\lambda}\left(B_{r}\left(\pi_{\lambda}\left(\pm1\omega\right)\right)\cap C_{\pm}\right)}{\mu_{\lambda}^{\pm}\left(B_{r}\left(\pi_{\lambda}\left(\pm1\omega\right)\right)\right)}=f_{\pm}\left(\pi_{\lambda}\left(\pm1\omega\right)\right)$$

On the other hand, for μ_{λ}^{+}-a.e. $x\in\mathbb{R}$ we have $x\in supp\left(\mu_{\lambda}^{+}\right)$ and thus:

$$\frac{\mu_{\lambda}\left(B_{r}\left(x\right)\right)}{\mu_{\lambda}^{+}\left(B_{r}\left(x\right)\right)}=\frac{\mu_{\lambda}^{-}\left(B_{r}\left(x\right)\right)+\mu_{\lambda}^{+}\left(B_{r}\left(x\right)\right)}{\mu_{\lambda}^{+}\left(B_{r}\left(x\right)\right)}\geq\frac{\mu_{\lambda}^{+}\left(B_{r}\left(x\right)\right)}{\mu_{\lambda}^{+}\left(B_{r}\left(x\right)\right)}=1$$

where we used the fact that:

$$x\in supp\left(\mu_{+}\right)\Longrightarrow\forall r>0\quad\mu_{\lambda}^{+}\left(B_{r}\left(x\right)\right)>0$$

Similarly for μ_{λ}^{-}-a.e. $x\in\mathbb{R}$. This amounts to the fact that for μ-a.e. $\omega\in\Omega$:

$$\lim_{r\to 0}\frac{\mu_{\lambda}\left(B_{r}\left(\pi_{\lambda}\left(\pm1\omega\right)\right)\right)}{\mu_{\lambda}^{\pm}\left(B_{r}\left(\pi_{\lambda}\left(\pm1\omega\right)\right)\right)}=f_{\pm}\left(\pi_{\lambda}\left(\pm1\omega\right)\right)\in[1,\infty)$$

So for μ-a.e. $\omega\in\Omega$:

$$f_{\pm}\left(\pi_{\lambda}\left(\pm1\omega\right)\right)\cdot D_{\alpha}^{+}\left(\mu_{\lambda}^{\pm},\pi_{\lambda}\left(\pm1\omega\right)\right)=f_{\pm}\left(\pi_{\lambda}\left(\pm1\omega\right)\right)\cdot\limsup_{r\to 0}\frac{\mu_{\lambda}^{\pm}\left(B_{r}(\pi_{\lambda}\left(\pm1\omega\right))\right)}{\left(2r\right)^{\alpha}}=$$

$$=\limsup_{r\to 0}\left(\frac{\mu_{\lambda}^{\pm}\left(B_{r}(\pi_{\lambda}\left(\pm1\omega\right))\right)}{\left(2r\right)^{\alpha}}\cdot\frac{\mu_{\lambda}\left(B_{r}\left(\pi_{\lambda}\left(\pm1\omega\right)\right)\right)}{\mu_{\lambda}^{\pm}\left(B_{r}\left(\pi_{\lambda}\left(\pm1\omega\right)\right)\right)}\right)=$$

$$=\limsup_{r\to 0}\frac{\mu_{\lambda}\left(B_{r}(\pi_{\lambda}\left(\pm1\omega\right))\right)}{\left(2r\right)^{\alpha}}=D_{\alpha}^{+}\left(\mu_{\lambda},\pi_{\lambda}\left(\pm1\omega\right)\right)$$

[9] These can only receive the value ∞ at a $\mu_{\lambda}^{\pm}$-null set at most (see theorem 2.12 in [5])

Leaving us with:

$$D_\alpha^+\left(\mu_\lambda^\pm, \pi_\lambda(\pm 1\omega)\right) = \frac{1}{f_\pm(\pi_\lambda(\pm 1\omega))} D_\alpha^+(\mu_\lambda, \pi_\lambda(\pm 1\omega)) \tag{6.5}$$

where $\frac{1}{f_\pm(x)} > 0$.
Taking (6.4) and 6.5 we conclude that for μ-a.e. $\omega \in \Omega$:

$$D_\alpha^+(\mu_\lambda, \pi_\lambda\omega) \geq \lambda^\alpha \cdot D_\alpha^+\left(\mu_\lambda^\pm, \pi_\lambda(\pm 1\omega)\right) = \frac{\lambda^\alpha}{f_\pm(\pi_\lambda(\pm 1\omega))} \cdot D_\alpha^+(\mu_\lambda, \pi_\lambda(\pm 1\omega))$$

and consequently for μ-a.e. $\omega \in A_\infty^\alpha$, $D_\alpha^+(\mu_\lambda, \pi_\lambda(\pm 1\omega)) < \infty$ or $\pm 1\omega \in A_\infty^\alpha$ amounting to $\sigma^{-1}(A_\infty^\alpha) \underset{\mu}{=} A_\infty^\alpha$ as required. ☐

Recall the following result from geometric measure theory (see 6.31 in [4]):

Theorem 6.4. *Let ν be a finite measure on $\mathbb{R}^d$ and $A \subseteq \mathbb{R}^d$. Then:*

$$\forall x \in A, \quad D_\alpha^+(\nu, x) > s \implies \mathcal{H}^\alpha(A) \leq \frac{C}{s} \cdot \nu(A) \tag{6.6}$$

where C is a constant depending only on d. And:

$$\forall x \in A, \quad D_\alpha^+(\nu, x) < t \implies \mathcal{H}^\alpha(A) \geq \frac{1}{2^\alpha t} \cdot \nu(A) \tag{6.7}$$

Proposition 6.5. *For any $\alpha \geq 0$ either $\mu_\lambda \ll \mathcal{H}^\alpha$ or $\mu_\lambda \perp \mathcal{H}^\alpha$.*

Proof. The set A_∞^α is measurable admitting $\sigma^{-1}(A_\infty^\alpha) \underset{\mu}{=} A_\infty^\alpha$ while μ is σ-ergodic hence $\mu(A_\infty^\alpha) \in \{0, 1\}$. We will view the two cases:
If $\mu(A_\infty^\alpha) = 0$, we know that for all $x \in \pi_\lambda(\Omega \backslash A_\infty^\alpha)$, $D_\alpha^+(\mu_\lambda, x) = \infty$ and in particular $D_\alpha^+(\mu, x) > s$ for any $s > 0$. Using (6.6) we deduce that for any s, $\mathcal{H}^\alpha(\pi_\lambda(\Omega \backslash A_\infty^\alpha)) \leq \frac{C}{s} \cdot \mu_\lambda(\pi_\lambda(\Omega \backslash A_\infty^\alpha))$ thus leading to $\mathcal{H}^\alpha(\pi_\lambda(\Omega \backslash A_\infty^\alpha)) = 0$ while $\mu_\lambda(\mathbb{R} \backslash \pi_\lambda(\Omega \backslash A_\infty^\alpha)) = \mu(A_\infty^\alpha) = 0$ and by definition $\mu_\lambda \perp \mathcal{H}^\alpha$.
If on the other hand $\mu(A_\infty^\alpha) = 1$, then we denote for each $1 \leq n \in \mathbb{N}$:

$$A_n^\alpha = \left\{\omega \in \Omega \,\middle|\, D_\alpha^+(\mu_\lambda, \pi_\lambda(\omega)) < n\right\}$$

Clearly $A_\infty^\alpha = \bigcup_{n=1}^\infty A_n^\alpha$ and $\lim_{n\to\infty} \mu(A_n^\alpha) = \mu(A_\infty^\alpha) = 1$. Let $E \subseteq \mathbb{R}$ be a set with $\mathcal{H}^\alpha(E) = 0$. For every $\varepsilon > 0$ there exists an n for which $\mu\left(\pi_\lambda^{-1} E \backslash A_n^\alpha\right) < \varepsilon$. We denote $E_n = E \cap \pi_\lambda A_n^\alpha$ and recall that for all $x \in E_n$, $D_\alpha^+(\mu_\lambda, x) < n$. Therefore by (6.7) we receive:

$$\mu_\lambda(E_n) \leq s^\alpha n \cdot \mathcal{H}^\alpha(E_n) \leq s^\alpha n \cdot \mathcal{H}^\alpha(E) = 0$$

This in turn implies:

$$\mu_\lambda(E) = \mu_\lambda(E \backslash E_n) = \mu\left(\pi_\lambda^{-1} E \backslash A_n^\alpha\right) < \varepsilon$$

Since this is true for any $\varepsilon > 0$ we conclude that $\mathcal{H}^{\alpha}(E) = 0 \implies \mu_{\lambda}(E) = 0$ or $\mu_{\lambda} \ll \mathcal{H}^{\alpha}$ as required.

□

6.II. **Proof of Measurability -** $D_{\mu}(\lambda, \omega, \tau)$**.**

Lemma 6.6. *The function* $D_{\mu}(\lambda, \omega, \tau) : I_h \times \Omega \times \Omega \to \mathbb{R}^{+} \cup \{\infty\}$ *is measurable.*

Proof. We will decompose $D_{\mu}(\lambda, \omega, \tau)$ and thus reduce the claim to a much simpler one. First we notice that for all $s > 0$ and $r_n \searrow s$ $B_{r_n}(x) \to B_s(x)$ and:

$$\frac{1}{2r_n}\mu_{\lambda}^{\mathscr{A}_E,\omega}\left(B_{r_n}(x)\right) \to \frac{1}{2s}\mu_{\lambda}^{\mathscr{A}_E,\omega}\left(B_s(x)\right)$$

Therefore:

$$D_{\mu}(\lambda, \omega, \tau) = \liminf_{\substack{q \to 0 \\ q \in \mathbb{Q} \cap (0,1)}} \frac{1}{2q}\mu_{\lambda}^{\mathscr{A}_E,\omega}\left(B_q(\pi_{\lambda}\tau)\right)$$

Taking lim inf preserves measurability hence we can reduce to proving the sequence of functions:

$$D_{\mu}^{q}(\lambda, \omega, \tau) = \mu_{\lambda}^{\mathscr{A}_E,\omega}\left(B_q(\pi_{\lambda}\tau)\right) = \mu_{\omega}^{\mathscr{A}_E}\left(\pi_{\lambda}^{-1}\left(B_q(\pi_{\lambda}\tau)\right)\right)$$

for $q \in \mathbb{Q} \cap (0,1)$ is measurable. By the increasing Martingale theorem, for any $B \in \mathscr{A}$:

$$\mu_{\omega}^{\mathscr{A}_E}(B) = \lim_{n\to\infty} \frac{\mu\left(B \cap [\omega]_{\mathscr{A}_E \cap \mathscr{A}_n}\right)}{\mu\left([\omega]_{\mathscr{A}_E \cap \mathscr{A}_n}\right)}$$

since $\mathscr{A}_E \cap \mathscr{A}_n \nearrow \mathscr{A}_E$. Hence we can reduce once more to proving the functions:

$$D_{\mu}^{q,n}(\lambda, \omega, \tau) = \sum_{\substack{A \in \mathscr{A}_E \cap \mathscr{A}_n \\ \mu(A) > 0}} \chi_A(\omega) \frac{\mu\left(\pi_{\lambda}^{-1}\left(B_q(\pi_{\lambda}\tau)\right) \cap A\right)}{\mu(A)}$$

It would suffice to show that given an $A \in \mathscr{A}_E \cap \mathscr{A}_n$ with $\mu(A) > 0$ the function:

$$D_{\mu}^{q,A}(\lambda, \tau) = \mu|_A\left(\pi_{\lambda}^{-1}\left(B_q(\pi_{\lambda}\tau)\right)\right) = \int \chi_{\pi_{\lambda}^{-1}\left(B_q(\pi_{\lambda}\tau) \cap \pi_{\lambda}(A)\right)}(\tau')\, d\mu(\tau')$$

is measurable.
Notice that for all λ, τ the set $B_q(\pi_{\lambda}\tau) \cap \pi_{\lambda}(A)$ is a finite union of intervals. Due to the continuity of π_{λ}, given a converging sequence $(\lambda_n, \tau_n) \to (\lambda_0, \tau_0)$, the set:

$$\left(\lim_{n\to\infty}\left(B_q(\pi_{\lambda_n}\tau_n) \cap \pi_{\lambda_n}(A)\right)\right) \Delta \left(B_q(\pi_{\lambda_0}\tau_0) \cap \pi_{\lambda_0}(A)\right) \subseteq \partial\left(B_q(\pi_{\lambda_0}\tau_0) \cap \pi_{\lambda_0}(A)\right)$$

is finite.
Using lemma 4.1, whenever $\lambda_0 \notin \mathcal{A}_{\{\pm1,0\}}$:

$$\mu\left(\pi_{\lambda_0}^{-1}\left(\partial\left(B_q(\pi_{\lambda_0}\tau) \cap \pi_{\lambda_0}(A)\right)\right)\right) = 0$$

meaning $\chi_{\pi_{\lambda_n}^{-1}\left(B_q(\pi_{\lambda_n}\tau_n) \cap \pi_{\lambda_n}(A)\right)}$ converges pointwise μ-a.e. to $\chi_{\pi_{\lambda_0}^{-1}\left(B_q\left(\pi_{\lambda_0}\tau_0\right) \cap \pi_{\lambda_0}(A)\right)}$ and thus by dominated convergence:

$$\int \chi_{\pi_{\lambda_n}^{-1}\left(B_q(\pi_{\lambda_n}\tau_n) \cap \pi_{\lambda_n}(A)\right)} d\mu \to \int \chi_{\pi_{\lambda_0}^{-1}\left(B_q\left(\pi_{\lambda_0}\tau_0\right) \cap \pi_{\lambda_0}(A)\right)} d\mu$$

This proves $D_{\mu}^{q,A}\Big|_{\left(\left(\frac{1}{2},1\right) \setminus \mathcal{A}_{\{\pm1,0\}}\right) \times \{\pm1\}^{\mathbb{N}}}$ is continuous. Having $\mathcal{L}\left(\mathcal{A}_{\{\pm1,0\}}\right) = 0$ concludes the proof. □

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